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2004 J. Phys. A: Math. Gen. 37 6221

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# Random walk generated by random permutations of $\{1, 2, 3, \dots, n + 1\}$

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Received 10 December 2003

Published 2 June 2004

Online at [stacks.iop.org/JPhysA/37/6221](http://stacks.iop.org/JPhysA/37/6221)

doi:10.1088/0305-4470/37/24/002

## Abstract

We study properties of a non-Markovian random walk  $X_l^{(n)}$ ,  $l = 0, 1, 2, \dots, n$ , evolving in discrete time  $l$  on a one-dimensional lattice of integers, whose moves to the right or to the left are prescribed by the rise-and-descent sequences characterizing random permutations  $\pi$  of  $[n + 1] = \{1, 2, 3, \dots, n + 1\}$ . We determine exactly the probability of finding the end-point  $X_n = X_n^{(n)}$  of the trajectory of such a permutation-generated random walk (PGRW) at site  $X$ , and show that in the limit  $n \rightarrow \infty$  it converges to a normal distribution with a smaller, compared to the conventional Pólya random walk, diffusion coefficient. We formulate, as well, an auxiliary stochastic process whose distribution is identical to the distribution of the intermediate points  $X_l^{(n)}$ ,  $l < n$ , which enables us to obtain the probability measure of different excursions and to define the asymptotic distribution of the number of ‘turns’ of the PGRW trajectories.

PACS numbers: 02.50.–r, 05.40.–a

## 1. Introduction

Properties of random unrestricted or certain patterns-avoiding permutations of  $[n] = \{1, 2, 3, \dots, n\}$  have been analysed by mathematicians in group theory and combinatorics for many years [1–3]. Studies of several problems emerging within this context, such as, e.g., the celebrated Ulam’s longest increasing subsequence problem (see, e.g., [4–6] and references therein), provided an entry to a rich and diverse circle of mathematical ideas [7], and were also found relevant to certain physical processes, including random surface growth [8–11] or 2D quantum gravity (see, e.g., [6]).

In this paper, we focus on random unrestricted<sup>4</sup> permutations from a bit different viewpoint addressing the following question: what sort of random walk one gets when random uniform permutations are used as a generator of the walk? Here we consider a simple model of such a permutation-generated random walk (PGRW), evolving in discrete time on a one-dimensional lattice of integers, in which model the moves of the walker to the right or to the left are prescribed by the rise-and-descent sequence characterizing each given permutation  $\pi = \{\pi_1, \pi_2, \pi_3, \dots, \pi_l, \dots, \pi_{n+1}\}$ . In a standard notation, the ‘rises’ (or the ‘descents’) of the permutation  $\pi$  are such values of  $l$  for which  $\pi_l < \pi_{l+1}$  ( $\pi_l > \pi_{l+1}$ ) [12]. We note that such a generator is evidently different from those producing conventional RWs, since here a finite amount  $n + 1$  of numbers is being shuffled and moreover, neither of any two numbers in each permutation may be equal to each other; this incurs, of course, correlations in the rise-and-descent sequences and implies that the resulting PGRW is a non-Markovian process.

In this paper, we determine exactly many characteristic properties of such a random walk. First, we define the probability  $\mathcal{P}_n(X)$  of finding the end-point  $X_n$  of the PGRW trajectory at site  $X$ . We show next that in the long-time limit  $\mathcal{P}_n(X)$  converges to a normal distribution in which the effective diffusion coefficient  $D = 1/6$  is three times smaller than the diffusion coefficient ( $D = 1/2$ ) of the conventional Pólya random walk in one dimension (1D), which signifies that correlations in the generator are marginally important. Indeed, assuming that correlations in the sequence of rises depend only on the relative distance between their positions, we deduce from  $\mathcal{P}_n(X)$  the two- and four-point correlation functions explicitly and demonstrate that correlations extend effectively to nearest neighbours only. Next, at a closer look on the intrinsic features of the PGRW trajectories, we formulate an auxiliary Markovian stochastic process. We show that, despite the fact that this process is Markovian, while the PGRW is not a Markovian process, the distribution of the auxiliary stochastic process appears to be identical to the distribution of the intermediate states of the PGRW trajectories. This enables us to obtain the probability measure of different excursions of the PGRW, deduce a general expression for  $k$ -point correlation functions in the rise-and-descent sequences, as well as to evaluate the asymptotic form of the distribution of the number of ‘turns’ of the PGRW trajectories, i.e. the number of times the walker changes the direction of its motion up to time  $n$ . Finally, we discuss, in the diffusion limit, the continuous space and time version of the PGRW.

The paper is outlined as follows: in section 2 we formulate the model more specifically. In section 3 we derive the distribution function of the end points of the walker’s trajectories exactly and analyse its asymptotical behaviour. In section 4 we consider the correlations in the rise-and-descent sequences and obtain explicit results for two- and four-point correlation functions. Further on, in section 5 we introduce an auxiliary stochastic process which has the same distribution as the intermediate points of the walkers trajectory  $X_l^{(n)}$ ,  $l = 1, 2, \dots, n - 1$ , and obtain the probability measure of different excursions. In section 6 we determine the asymptotic form of the distribution function of the number of ‘turns’ of the PGRW trajectory, relating it to  $k$ -point correlation functions in the rise-and-descent sequences. Next, in section 7 we discuss the continuous space and time analogue of the PGRW. Finally, in section 8 we conclude with a brief summary of our results and discussion.

## 2. Model

Let  $\pi = \{\pi_1, \pi_2, \pi_3, \dots, \pi_{n+1}\}$  denote a random, unconstrained permutation of  $[n + 1]$ . We rewrite it next in two-line notation as

<sup>4</sup> In the sense that all patterns are permitted.

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n + 1 \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_{n+1} \end{pmatrix} \tag{1}$$

where the numbers in the first line will be thought of as the values attained by a discrete ‘time’ variable  $l$ . We call, in a standard notation, as the ‘rise’ (or the ‘descent’) of the permutation  $\pi$ , such values of  $l$  for which  $\pi_l < \pi_{l+1}$  ( $\pi_l > \pi_{l+1}$ ) [12].

Consider now a walk evolving in time  $l$  on an infinite in both directions one-dimensional lattice of integers according to the following rules:

- at time moment  $l = 0$  the walker is placed at the origin;
- at time moment  $l = 1$  the walker is moved one step to the right if  $\pi_1 < \pi_2$ , i.e.,  $l = 1$  is a rise, or to the left if  $\pi_1 > \pi_2$ , i.e.  $l = 1$  is a descent;
- at time moment  $l = 2$ , the walker is moved one step to the right (left) if  $\pi_2 < \pi_3$  ( $\pi_2 > \pi_3$ , resp.) and etc.

Repeated  $l$  times, this procedure results in a random, permutation-dependent trajectory  $X_l^{(n)}$ , ( $l = 1, 2, \dots, n$ ),

$$X_l^{(n)} = \sum_{k=1}^l s_k \tag{2}$$

where the ‘spin’ variable  $s_k$  is

$$s_k = \theta(\pi_{k+1} - \pi_k) \tag{3}$$

$\theta(x)$  being the theta function with the properties

$$\theta(x) \equiv \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{otherwise.} \end{cases} \tag{4}$$

The questions, which we address here, are (a) the form of the probability  $\mathcal{P}_n(X)$  of finding such a walker at position  $X$  at time moment  $n$ , (b) asymptotical behaviour of  $\mathcal{P}_n(X)$ , (c) the probability measure of different excursions  $X_l^{(n)}$  and (d) the distribution function of the number of ‘turns’ of the PGRW trajectories. As a by-product of our analysis, we also determine some specific correlations in permutations, embodied in the moments of the walker’s trajectories., i.e. the  $k$ -point correlation functions in the rise-and-descent sequences, as well as the asymptotic distribution of the number of peaks and throughs in random permutations.

### 3. The probability distribution

One possible way to determine  $\mathcal{P}_n(X)$  is to reconstruct it from the moments  $\langle X_n^{2q} \rangle$  ( $q = 1, 2, 3, \dots$ ) of the end-point  $X_n$  of the walker’s trajectory,  $X_n = X_{l=n}^{(n)} = \sum_{k=1}^n s_k$ ,

$$\langle X_n^{2q} \rangle = \frac{1}{(n + 1)!} \sum_{\{\pi\}} \left[ \sum_{l=1}^n s_l \right]^{2q} \tag{5}$$

where the angle brackets denote ‘averaging’ procedure—summation with respect to the set  $\{\pi\}$  of all possible permutations  $\pi$ , weighted by their total number  $(n + 1)!$ .

Pursuing this kind of approach, we may then represent the summation over  $\{\pi\}$  as a multiple summation over states of a set of independent variables  $\{a_i\}$ , each running from 1 to  $n + 1$ :

$$\sum_{\{\pi\}} \cdots = \lim_{J \rightarrow \infty} \sum_{a_1=1}^{n+1} \sum_{a_2=1}^{n+1} \cdots \sum_{a_{n+1}=1}^{n+1} \exp \left( -J \sum_{(i,j)} \delta_{a_i, a_j} \right) \cdots \tag{6}$$

where the sum with the subscript  $(i, j)$  extends over all  $i, j$  pairs such that  $i, j \in 1, 2, 3, \dots, n+1$ , excluding  $= j$ ,  $\delta_{a_i, a_j}$  is the Kronecker-symbol, while the factor

$$\lim_{J \rightarrow \infty} \exp \left( -J \sum_{(i,j)} \delta_{a_i, a_j} \right) = \prod_{(i,j)} (1 - \delta_{a_i, a_j}) \quad (7)$$

accounts for the constraint that  $\pi_i \neq \pi_j$  for any  $i$  and  $j$ ,  $i \neq j$ .

Therefore, calculation of moments of  $X_n$  amounts to the evaluation of (rather) special correlations in an  $(n+1)$ -state Potts-like model with long-range interactions on the 1D lattice containing  $n+1$  sites, which may be approached, in particular, using a non-trivial transformation of variables proposed in [13]. It appears, however, that within this approach the calculation of already the second moment  $\langle X_n^2 \rangle$  is quite laborious, although the final result is simple.

On the other hand,  $\mathcal{P}_n(X)$  can be obtained straightforwardly, in an explicit form, following a different way of reasonings. To do this, let us recall that according to the definition of the PGRW, at the  $l$ th step the walker makes a move to the right (left) if  $l$  is the rise (descent) of the random permutation  $\pi$ . Now, let  $\mathcal{N}_\uparrow$  ( $\mathcal{N}_\downarrow$ ) be the number of ‘risers’ (‘descents’) in a given random permutation  $\pi$ . Then, for this given permutation  $\pi$  the end-point  $X_n$  of the walker’s trajectory is just

$$X_n = \mathcal{N}_\uparrow - \mathcal{N}_\downarrow \quad (8)$$

which can be rewritten, using an evident ‘conservation law’  $\mathcal{N}_\uparrow + \mathcal{N}_\downarrow = n$  as

$$X_n = 2\mathcal{N}_\uparrow - n. \quad (9)$$

Therefore, for a given permutation  $\pi$ , the end-point  $X_n$  of the walker’s trajectory  $X_l^{(n)}$  is entirely *fixed* by the value of the number of *risers* in this permutation.

Now, a total number of permutations  $\pi$  of  $[n+1]$  having *exactly*  $\mathcal{N}_\uparrow$  risers is given by the so-called Eulerian number [14]:

$$\left\langle \begin{matrix} n+1 \\ \mathcal{N}_\uparrow \end{matrix} \right\rangle = \sum_{r=0}^{\mathcal{N}_\uparrow+1} (-1)^r \binom{n+2}{r} (\mathcal{N}_\uparrow + 1 - r)^{n+1} \quad (10)$$

where  $\binom{a}{b}$  denotes the binomial coefficient. Consequently, in virtue of equation (9), we have that the probability  $\mathcal{P}_n(X)$  of finding the walker at position  $X$  at time  $n$  ( $-n \leq X \leq n$ ) is given by

$$\mathcal{P}_n(X) = \frac{[1 + (-1)^{X+n}]}{2(n+1)!} \left\langle \begin{matrix} n+1 \\ \frac{X+n}{2} \end{matrix} \right\rangle. \quad (11)$$

Using the representation of the Eulerian number in equation (10), the result in equation (11) can be also written down in explicit form as the following finite series:

$$\mathcal{P}_n(X) = \frac{[1 + (-1)^{X+n}]}{2(n+1)!} \sum_{r=0}^{(X+n+2)/2} (-1)^r \binom{n+2}{r} \left( \frac{X+n+2}{2} - r \right)^{n+1}. \quad (12)$$

We derive next several useful integral representations of the distribution function  $\mathcal{P}_n(X)$  and of the corresponding lattice Green function [15], which will allow for an easy access to the large- $n$  asymptotical behaviour of  $\mathcal{P}_n(X)$ .

Making use of the equality [16]:

$$\frac{1}{(n+1)!} \sum_{r=0}^{[(n+2+b)/2]} (-1)^r \binom{n+2}{r} \left( \frac{n+2+b}{2} - r \right)^{n+1} = \frac{2}{\pi} \int_0^\infty \left( \frac{\sin(k)}{k} \right)^{n+2} \cos(bk) dk \quad (13)$$

we find that  $\mathcal{P}_n(X)$  admits the following compact form

$$\mathcal{P}_n(X) = \frac{[1 + (-1)^{X+n}]}{\pi} \int_0^\infty \left(\frac{\sin(k)}{k}\right)^{n+2} \cos(Xk) dk. \tag{14}$$

Note that the integral representation in equation (14) is different from the usually encountered forms since here the upper terminal of integration is infinity. As a matter of fact, an integral representation of  $\mathcal{P}_n(X)$  in equation (11), in which the integration extends over the first Brillouin zone only, has a completely different form, compared to that in equation (14), and reads

$$\begin{aligned} \mathcal{P}_n(X) &= \frac{(-1)^{n+1}}{(n+1)! \pi} \int_0^\pi \left( \sin^{n+2}(k) \frac{d^{n+1}}{dk^{n+1}} \cot(k) \right) \cos(Xk) dk \\ &= \frac{1}{\pi^{n+3} (n+1)!} \int_0^\pi \left[ \Psi_{n+1} \left( 1 - \frac{k}{\pi} \right) + (-1)^n \Psi_{n+1} \left( \frac{k}{\pi} \right) \right] \sin^{n+2}(k) \cos(Xk) dk \end{aligned} \tag{15}$$

where  $\Psi_n(k)$  is the polygamma function [16]. Derivation of the result in equation (15) is outlined in appendix A.

Consider finally the asymptotic forms of the probability distribution function. To do this, it is expedient to turn to the lattice Green function  $\mathcal{G}(X, z)$  associated with the distribution function of the end point of the PGRW trajectories. From equations (14) and (15) we find that the lattice Green function of the end point of the PGRW is given by

$$\begin{aligned} \mathcal{G}(X, z) &= \sum_{n=0}^\infty \mathcal{P}_n(X) z^n = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin(k)}{k}\right)^2 \frac{\cos(Xk) dk}{(1 - z^2 \frac{\sin^2(k)}{k^2})} \\ &= \frac{1}{\pi z} \int_0^\pi \frac{\sin(z \sin(k)) \cos(Xk)}{\sin(k - z \sin(k))} dk. \end{aligned} \tag{16}$$

Now, in the limit  $z \rightarrow 1^-$  (which corresponds to the large- $n$  behaviour of the distribution function  $\mathcal{P}_n(X)$ ), one finds that the leading behaviour of  $\mathcal{G}(X, z)$  is as follows:

$$\mathcal{G}(X, z \rightarrow 1^-) \sim \sqrt{\frac{3}{2(1-z)}} \exp(-\sqrt{6(1-z)}|X|) \tag{17}$$

which implies that in the asymptotic limit  $n \rightarrow \infty$ , the probability  $\mathcal{P}_n(X)$  of finding the end point of the walker’s trajectory at position  $X$  converges to a normal distribution:

$$\mathcal{P}_n(X) \sim \left(\frac{3}{2\pi n}\right)^{1/2} \exp\left(-\frac{3X^2}{2n}\right). \tag{18}$$

Therefore, in the limit  $n \rightarrow \infty$  the correlations in the generator of the walk, random permutations of  $[n + 1]$ , appear to be marginally important; that is, they do not result in *anomalous* diffusion, but merely affect the ‘diffusion coefficient’ making it three times smaller than the diffusion coefficient of the standard 1D Pólya walk. We will consider the form of correlations in the following section.

#### 4. Correlations in the rise-and-descent sequence

We address now a somehow ‘inverse’ problem: that is, given the distribution  $\mathcal{P}_n(X)$ , we aim to determine two- and four-point correlations in the rise-and-descent sequences.

We start with the analysis of two-point correlations. From equation (5) we represent the second moment of the walker’s displacement as

$$\langle X_n^2 \rangle = \langle (\mathcal{N}_\uparrow - \mathcal{N}_\downarrow)^2 \rangle = \left\langle \left[ \sum_{l=1}^n s_l \right]^2 \right\rangle = n + 2 \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^n C_{j_1, j_2}^{(2)} \tag{19}$$

where we denote as  $C_{j_1, j_2}^{(2)}$  the two-point correlation function of the rise-and-descent sequence,

$$C_{j_1, j_2}^{(2)} = \langle s_{j_1} s_{j_2} \rangle. \quad (20)$$

From our previous analysis, we know already that  $\langle X_n^2 \rangle \sim n/3$  as  $n \rightarrow \infty$ , (see equation (18)). Consequently, in this limit the sum on the rhs of equation (19) is expected to behave as

$$\sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^n C_{j_1, j_2}^{(2)} \sim -\frac{n}{3} \quad (21)$$

where the sign ‘-’ signifies that the *rise-and-descent* sequence is predominantly *anticorrelated* and the probability of having two neighbouring rises (or two descents) is lower than the probability of having a rise neighbouring to a descent. The question now is how do the correlations decay with a relative distance  $m = |j_2 - j_1|$ ? We will answer this question here assuming that  $C_{j_1, j_2}^{(2)}$  is a function of the distance  $|j_2 - j_1|$  only, i.e.

$$C_{j_1, j_2}^{(2)} \equiv C_{j_2, j_1}^{(2)} = C^{(2)}(m = |j_2 - j_1|). \quad (22)$$

We note that such an assumption seems quite plausible at the first glance, since the rises (and descents) are evidently uniformly distributed on the interval  $[1, n]$ . We will show in the following section that it is actually the case using an auxiliary stochastic process having the same distribution of the intermediate steps as the PGRW.

Introducing the generating function of the form

$$\mathcal{X}^{(2)}(z) = \sum_{n=0}^{\infty} \langle X_n^2 \rangle z^n \quad (23)$$

we get from equation (19) the following relation:

$$\mathcal{X}^{(2)}(z) = \frac{z}{(1-z)^2} + \frac{2z}{(1-z)^2} C^{(2)}(z) \quad (24)$$

where  $C^{(2)}(z)$  is the generating function of two-point correlations:

$$C^{(2)}(z) = \sum_{m=1}^{\infty} z^m C^{(2)}(m). \quad (25)$$

Consequently, in order to define  $C^{(2)}(m)$ , we have to evaluate  $\mathcal{X}^{(2)}(z)$ . To do this, we proceed as follows: define the Fourier-transformed  $\mathcal{P}_n(X)$  as

$$\tilde{\mathcal{P}}_n(k) = \sum_{X=-n}^n \exp(ikX) \mathcal{P}_n(X). \quad (26)$$

Noting first that  $\mathcal{P}_n(n+1) \equiv 0$ , then, making use of the representation in equation (11) and changing the summation variable, we get

$$\tilde{\mathcal{P}}_n(k) = \frac{\exp(-ikn)}{(n+1)!} \sum_{r=0}^{n+1} \exp(2ikr) \binom{n+1}{r}. \quad (27)$$

Further on, using the equality

$$\sum_{r=0}^{n+1} y^{n+1-r} \binom{n+1}{r} = (1-y)^{n+2} \text{Li}_{-n-1}(y) \quad (28)$$

where  $\text{Li}_{-n-1}(y)$  denotes the polylogarithm function [16], we obtain, by setting  $y = \exp(-2ik)$ ,

$$\tilde{\mathcal{P}}_n(k) = \frac{(2i)^{n+2} \sin^{n+2}(k)}{(n+1)!} \text{Li}_{-n-1}(\exp(-2ik)). \quad (29)$$

Next, by taking advantage of the expansion [16]

$$\text{Li}_{-n-1}(y) = (n + 1)! \left(\ln \frac{1}{y}\right)^{-n-2} - \sum_{j=0}^{\infty} \frac{B_{n+2+j}}{j!(n + 2 + j)} (\ln y)^j \tag{30}$$

where  $B_j$  stands for the Bernoulli numbers, we arrive at the following result:

$$\tilde{\mathcal{P}}_n(k) = \left(\frac{\sin(k)}{k}\right)^{n+2} \left[ 1 - \begin{cases} j_1(k) & \text{for } n \text{ odd} \\ j_2(k) & \text{for } n \text{ even} \end{cases} \right] \tag{31}$$

with

$$j_1(k) = \frac{(-1)^{(n+1)/2}(2k)^{n+3}}{(n + 1)!} \sum_{j=0}^{\infty} \frac{(-1)^j B_{n+3+2j}}{(2j + 1)!(n + 3 + 2j)} (2k)^{2j} \tag{32}$$

and

$$j_2(k) = -\frac{(-1)^{n/2}(2k)^{n+2}}{(n + 1)!} \sum_{j=0}^{\infty} \frac{(-1)^j B_{n+2+2j}}{(2j)!(n + 2 + 2j)} (2k)^{2j}. \tag{33}$$

Now, we aim to determine  $\langle X_n^2 \rangle$  and  $\langle X_n^4 \rangle$  explicitly. Expanding  $\tilde{\mathcal{P}}_n(k)$  defined by equations (31), (32) and (33) into the Taylor series in powers of  $k$ , we find that  $\tilde{\mathcal{P}}_n(k)$  obeys

$$\begin{aligned} \tilde{\mathcal{P}}_n(k) = & 1 - \frac{1}{6}(n + 2 - 2\delta_{n,0})k^2 + \frac{1}{360}((5n + 8)(n + 2) \\ & - 16\delta_{n,0} - 24\delta_{n,1} + 8\delta_{n,2})k^4 + \mathcal{O}(k^6) \end{aligned} \tag{34}$$

which yields

$$\langle X_n^2 \rangle \equiv \frac{1}{3}(n + 2 - 2\delta_{n,0}) \tag{35}$$

and

$$\langle X_n^4 \rangle \equiv \frac{1}{15}((5n + 8)(n + 2) - 16\delta_{n,0} - 24\delta_{n,1} + 8\delta_{n,2}). \tag{36}$$

We note parenthetically that the obtained distribution  $\mathcal{P}_n(X)$  in equation (11) appears to be platykurtic, since the kurtosis excess

$$\gamma = \frac{\langle X_n^4 \rangle}{\langle X_n^2 \rangle^2} - 3 = -\frac{6}{5n} + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{37}$$

is negative.

Now, from equation (35) we find

$$\mathcal{X}^{(2)}(z) = \frac{z(3 - 2z)}{3(1 - z)^2} \tag{38}$$

which implies that  $\mathcal{C}^{(2)}(z) = -z/3$  and hence, that

$$\mathcal{C}^{(2)}(m) \equiv \begin{cases} -\frac{1}{3} & \text{if } m = 1 \\ 0 & \text{if } m \geq 2. \end{cases} \tag{39}$$

Note that equation (39) signifies that for  $m \geq 2$  the two-point correlations  $\mathcal{C}^{(2)}(m = |j_2 - j_1|)$  in the rise-and-descent sequence decouple into the product  $\langle s_{j_1} \rangle \langle s_{j_2} \rangle$  and hence, vanish. Consequently, two-point correlations are short-ranged extending to the nearest neighbours only. Therefore, the probability  $p_{\uparrow, \uparrow}(m)$  of having two rises (or two descents) a distance  $m$  apart of each other is

$$p_{\uparrow, \uparrow}(m) = p_{\downarrow, \downarrow}(m) \equiv \frac{1 + \mathcal{C}^{(2)}(m)}{4} = \begin{cases} \frac{1}{6} & \text{if } m = 1 \\ \frac{1}{4} & \text{if } m \geq 2 \end{cases} \tag{40}$$



while

$$p_{\uparrow,\downarrow}(m) = p_{\downarrow,\uparrow}(m) \equiv \frac{1 - C^{(2)}(m)}{4} = \begin{cases} \frac{1}{3} & \text{if } m = 1 \\ \frac{1}{4} & \text{if } m \geq 2. \end{cases} \quad (41)$$

Consider next behaviour of the four-point correlations:

$$\mathcal{C}_{j_1, j_2, j_3, j_4}^{(4)} = \langle s_{j_1} s_{j_2} s_{j_3} s_{j_4} \rangle \quad (42)$$

where  $j_1 < j_2 < j_3 < j_4$ . From our result in equation (39) it seems natural to expect that  $\mathcal{C}_{j_1, j_2, j_3, j_4}^{(4)}$  vanishes as soon either (or both)  $j_2 - j_1$  or  $j_4 - j_3$  are greater than unity. Consequently, only non-vanishing four-point correlations are of the form

$$\mathcal{C}^{(4)}(m) = \langle s_{j_1} s_{j_1+1} s_{j_1+m+1} s_{j_1+m+2} \rangle. \quad (43)$$

To evaluate  $\mathcal{C}^{(4)}(m)$ , we will proceed along the same lines as we did with the analysis of two-point correlations. From equation (5) we have

$$\begin{aligned} \langle X_n^4 \rangle &= \langle (\mathcal{N}_\uparrow - \mathcal{N}_\downarrow)^4 \rangle = \left\langle \left[ \sum_{l=1}^n s_l \right]^4 \right\rangle \\ &= 4! \sum \frac{\langle s_1^{m_1} s_2^{m_2} \dots s_n^{m_n} \rangle}{m_1! m_2! \dots m_n!} \end{aligned} \quad (44)$$

where the sum extends over all positive integer solutions of equation  $m_1 + m_2 + \dots + m_n = 4$ . Taking into account our results for  $\mathcal{C}^{(2)}(m)$ , as well as noting that  $s_l^4 = s_l^2 \equiv 1$ , while  $s_l^3 = s_l$  for any  $l$ , we find that

$$\begin{aligned} \langle X_n^4 \rangle &= \frac{4!}{4!} n + \frac{4!}{2!2!} \frac{n(n-1)}{2} \theta(n-2) + 2 \frac{4!}{3!1!} (n-1) \mathcal{C}^{(2)}(1) \theta(n-2) \\ &\quad + \frac{4!}{2!(1!)^2} (n(n-3) + 2) \mathcal{C}^{(2)}(1) \theta(n-3) + 4! \sum_{m=1}^{n-3} (n-2-m) \mathcal{C}^{(4)}(m) \theta(n-4). \end{aligned} \quad (45)$$

Multiplying both sides of the last equation by  $z^n$  and performing summation, we find the following relation between the generating function  $\mathcal{X}^{(4)}(z)$  of the fourth moment of the walker's displacement and the generating function  $\mathcal{C}^{(4)}(z)$  of the four-point correlations:

$$\mathcal{X}^{(4)}(z) = \frac{15z + 35z^2 - 80z^3}{15(1-z)^3} + \frac{4!z^3}{(1-z)^2} \mathcal{C}^{(4)}(z). \quad (46)$$

On the other hand, from equation (36) we have that

$$\mathcal{X}^{(4)}(z) = \frac{z(35z + 15 - 80z^2 + 48z^3 - 8z^4)}{15(1-z)^3} \quad (47)$$

which yields, eventually

$$\mathcal{C}^{(4)}(z) = \frac{z(6-z)}{45(1-z)} = \frac{2}{15}z + \frac{1}{9}(z^2 + z^3 + z^4 + \dots) \quad (48)$$

and hence

$$\mathcal{C}^{(4)}(m) = \begin{cases} \frac{2}{15} & \text{if } m = 1 \\ \frac{1}{9} & \text{if } m \geq 2. \end{cases} \quad (49)$$

Note that equation (49) signifies that the four-point correlations decouple into the product of nearest-neighbour two-point correlations,  $\mathcal{C}^{(4)}(m) = \mathcal{C}^{(2)}(1)\mathcal{C}^{(2)}(1)$ , for  $m \geq 2$ .

Finally, noting that three-point correlations of the form  $\langle s_j s_{j+1} s_{j+2} \rangle$  are equal to zero, we may straightforwardly calculate the probabilities of several particular configurations involving three and four rises and descents. In particular, the probability of having two neighbouring rises and a descent at distance  $m$  apart of them is given by

$$\begin{aligned}
 p_{\uparrow\uparrow,\downarrow}(m) &= \frac{1}{8} \langle (1 + s_j)(1 + s_{j+1})(1 - s_{j+m+1}) \rangle \\
 &= \begin{cases} \frac{1}{8} & \text{if } m = 1 \\ \frac{1}{12} & \text{if } m \geq 2 \end{cases} \tag{50}
 \end{aligned}$$

while the probability of having two neighbouring rises and another rise at distance  $m$  apart of them obeys

$$\begin{aligned}
 p_{\uparrow\uparrow,\uparrow}(m) &= \frac{1}{8} \langle (1 + s_j)(1 + s_{j+1})(1 + s_{j+m+1}) \rangle \\
 &= \begin{cases} \frac{1}{24} & \text{if } m = 1 \\ \frac{1}{12} & \text{if } m \geq 2. \end{cases} \tag{51}
 \end{aligned}$$

In a similar fashion, we get that the probability of having two rises and another pair of rises (descents) at distance  $m$  apart of them is given by

$$p_{\uparrow\uparrow,\uparrow\uparrow}(m) = \begin{cases} \frac{1}{120} & \text{if } m = 1 \\ \frac{1}{36} & \text{if } m \geq 2 \end{cases} \tag{52}$$

the corresponding probabilities involving three rises obey

$$\begin{aligned}
 p_{\uparrow\uparrow,\downarrow\uparrow}(m) &= p_{\uparrow\downarrow,\uparrow\uparrow}(m) = \begin{cases} \frac{3}{40} & \text{if } m = 1 \\ \frac{1}{18} & \text{if } m \geq 2 \end{cases} \\
 p_{\uparrow\uparrow,\uparrow\downarrow}(m) &= p_{\downarrow\uparrow,\uparrow\uparrow}(m) = \begin{cases} \frac{1}{30} & \text{if } m = 1 \\ \frac{1}{18} & \text{if } m \geq 2 \end{cases} \tag{53}
 \end{aligned}$$

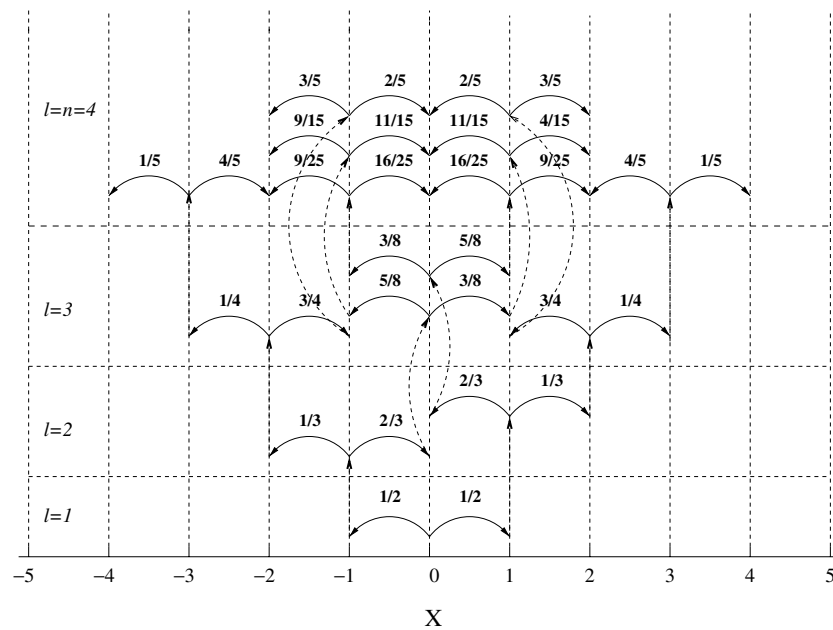
while the probabilities involving two rises and two descents follow

$$\begin{aligned}
 p_{\uparrow\downarrow,\uparrow\downarrow}(m) &= \begin{cases} \frac{2}{15} & \text{if } m = 1 \\ \frac{1}{9} & \text{if } m \geq 2 \end{cases} & p_{\uparrow\uparrow,\downarrow\downarrow}(m) &= \begin{cases} \frac{1}{20} & \text{if } m = 1 \\ \frac{1}{36} & \text{if } m \geq 2 \end{cases} \\
 p_{\downarrow\uparrow,\uparrow\downarrow}(m) &= \begin{cases} \frac{11}{120} & \text{if } m = 1 \\ \frac{1}{9} & \text{if } m \geq 2. \end{cases} \tag{54}
 \end{aligned}$$

Equations (50), (51) to (54) show explicitly that the probabilities of different configurations of the rise-and-descent sequences depend not only on the number of rises, but also on their order within the sequence.

### 5. Trajectories $X_l^{(n)}$ for $l < n$

So far, we have defined the distribution of the end points of the trajectories  $X_l^{(n)}$ , but, apart from the explicit results on the form of two- and four-point correlations, which allow us to reconstruct trajectories  $X_l^{(n)}$  with  $n = 4$  (see figure 1), we do not have an access to information on the distribution of the intermediate points  $l = 1, 2, 3, \dots, n - 1$ . On the other hand, it seems to be quite non-trivial. Indeed, we deal with a random walk, which makes a move of unit length at each moment of time with probability 1, but nonetheless loses somehow two thirds of the diffusion coefficient. Given that the correlations are short-ranged and extend to nearest



**Figure 1.** A set of all possible PGRW trajectories  $X_l^{(n)}$  for  $n = 4$ . Numbers above the solid arcs with arrows indicate the corresponding transition probabilities. Dashed lines with arrows connect the trajectories for different values of  $l$ .

neighbours only, one might expect that the PGRW behaves effectively as an ‘antipersistent’ RW [15] with a short-range one-step memory, such that the probability, in view of the results in equations (40) and (41), of making on the  $l$ th step a move in the same direction at which it made a move on the  $(l - 1)$ th step is  $1/3$ , while the probability of changing the direction is  $2/3$ . One readily verifies that for such a walk  $\langle X_n^2 \rangle \sim n/2$ , i.e. the reduction in the ‘diffusion coefficient’ is smaller than the one we actually find for the PGRW, ( $\langle X_n^2 \rangle \sim n/3$ ). As a matter of fact, as we observe in figure 1, for the PGRW the memory of the ‘antipersistence’ is stronger and depends not only on the number of steps to the right or to the left, which the walker has already made, but also on their order. In other words, the PGRW represents a genuine non-Markovian process.

### 5.1. The probability distribution of $X_l^{(n)}$ for $l < n$ .

To determine the structure of excursions  $X_l^{(n)}$  of the PGRW we adapt a method proposed by Hammersley [17] in his analysis of the evolution of the longest increasing subsequence, and elucidated subsequently in [18]. The basic idea behind this approach, which we exploit here, is to build recursively an auxiliary *Markovian* stochastic process  $Y_l$ , which is distributed exactly as  $X_l^{(n)}$ .

At each time step  $l$ , let us define a real-valued random variable  $x_{l+l}$ , uniformly distributed in  $[0, 1]$ . Consider next a random walk on an infinite in both directions one-dimensional lattice of integers whose trajectory  $Y_l$  is constructed according to the following step-by-step process: at each time moment  $l$  a point-like particle is created at position  $x_{l+l}$ . If  $x_{l+1} > x_l$ , a walker

is moved one step to the right; otherwise, it is moved one step to the left. The trajectory  $Y_l$  is then given by

$$Y_l = \sum_{k=1}^l \theta(x_{k+1} - x_k) \tag{55}$$

where  $\theta(x)$  is the theta function defined in equation (4). We note that the joint process  $(x_{l+1}, Y_l)$ , and therefore  $Y_l$ , are *Markovian* since they depend only on  $(x_l, Y_{l-1})$ . Note also that  $Y_l$  is the sum of *correlated* random variables; hence, one has to be cautious when applying central limit theorems. A central limit theorem indeed holds for the Markovian process  $Y_l$ , but the summation rule for the variance is not valid.

Now, we aim at proving that the probability  $\mathcal{P}(Y_l = Y)$  that at time moment  $l$  the trajectory  $Y_l$  of the auxiliary Markovian process appears on the site  $Y$  has exactly the same form as the Eulerian distribution obtained for the end point of the trajectory  $X_l^{(n)}$ . That is,  $\mathcal{P}(Y_l = Y)$  obeys

$$\mathcal{P}(Y_l = Y) = \frac{[1 + (-1)^{Y+l}]}{2(l + 1)!} \left\langle \frac{Y+l}{2} \right\rangle. \tag{56}$$

To prove this statement, we assign a random permutation  $\pi$  of  $[n + 1]$  to each realization of sequence  $\{x_l\}$ ,  $l = 1, 2, \dots, n + 1$ , by ordering the  $x_l$  and requiring that  $\pi_l$  is the row of  $x_l$ :

$$x_{\pi_1} < x_{\pi_2} < \dots < x_{\pi_{n+1}}. \tag{57}$$

Next, we note that according to the definition in equation (55) each random trajectory  $Y_l$ ,  $l \leq n$ , is entirely determined by ordering of the corresponding sequence  $\{x_i\}_{i \leq l+1}$ . This implies that  $Y_l$  is adequately determined by the permutation  $\pi$  defined by such an ordering procedure.

One writes next the weight  $p(Y_l)$  of a given trajectory  $Y_l$  as an integral over the sequences  $\{x_i\}_{i \leq l+1}$  generating  $Y_l$ , which can be represented as a sum over all corresponding permutations:

$$p(Y_l) = \int_{\{x_i\} \text{ generates } Y_l} dx_1 \dots dx_{n+1} = \sum_{\pi \text{ generates } Y_l} p(\pi) \tag{58}$$

where  $p(\pi)$  is the probability of a given random permutation  $\pi$ . On the other hand,  $p(\pi)$  obeys

$$p(\pi) = \int_{x_{\pi^{-1}[\pi(k)-1]} \leq x_k \leq x_{\pi^{-1}[\pi(k)+1]}} dx_1 \dots dx_{n+1} = \int_0^1 dx_{n+1} \int_0^{x_{n+1}} dx_n \dots \int_0^{x_2} dx_1 = \frac{1}{(n + 1)!} \tag{59}$$

where  $x_{\pi^{-1}(0)} = 0$ ,  $x_{\pi^{-1}(n+2)} = 1$  and  $\pi^{-1}$  denotes the functional inverse of  $\pi$ , such that  $\pi^{-1}(\pi_k) = k$ . Finally, noting that integration over the variables  $x_{i > l+1}$  obviously gives 1, in view of the purely iterative definition of the process  $Y_l$ , which is independent of the ‘future’  $x_{i > l+1}$ , we have

$$\sum_{\pi \text{ generates } Y_l} p(\pi) = \sum_{\pi \text{ generates } Y_l} \frac{1}{(n + 1)!} \tag{60}$$

which is precisely the probability of the trajectory  $X_l^{(n)}$  with  $l \leq n$ .

Therefore,  $Y_l$  and  $X_l^{(n)}$  are identically distributed. We emphasize that the distribution of  $Y_l$ , and therefore the distribution of  $X_l^{(n)}$ , do *not* depend on  $n$  for  $l < n$ , which is a rather counter-intuitive result for the process  $X_l^{(n)}$ . Consequently, the probability  $\mathcal{P}(X_l^{(n)} = X)$  that at any intermediate step  $l$ ,  $l = 1, 2, 3, \dots, n - 1$ , the trajectory  $X_l^{(n)}$  appears at the site  $X$  obeys

$$\mathcal{P}(X_l^{(n)} = X) = \mathcal{P}_l(X) \tag{61}$$

where  $\mathcal{P}_l(X)$  is defined by equation (11).

We note here parenthetically that the result in equation (61) allows us to study the properties of the random process  $X_l^{(n)}$  which depend on the intermediate states, such as, e.g., the span, the maximal excursions within time  $n$ , the time spent on positive sites and so on. In particular, it might be instructive to compare the growth of the average length  $L_n$  of the longest subsequence of (not necessarily consecutive) rises, i.e. of the so-called ‘longest increasing subsequence’, and the growth of the average maximal positive excursion  $\langle \max\{X_n\} \rangle$  of the PGRW, which is also supported by rises in random permutations. Since we realized that the PGRW converges in distribution to standard Pólya walk with diffusion coefficient  $D = 1/6$ , we may estimate  $\langle \max\{X_n\} \rangle$  as [15]:

$$\langle \max\{X_n\} \rangle \sim \left(\frac{4D}{\pi}\right)^{1/2} \sqrt{n} = \left(\frac{2}{3\pi}\right)^{1/2} \sqrt{n} \approx 0.461 \sqrt{n}. \quad (62)$$

Hence, the growth of the maximal positive displacement of the PGRW proceeds at a slower rate than that of  $L_n$ ,  $L_n \sim 2\sqrt{n}$ , due to a numerical factor, which is more than four times smaller.

### 5.2. Probability measure of different trajectories

The equivalence of the processes  $Y_l$  and  $X_l^{(n)}$  allows us to determine explicitly the probability of any given trajectory. We note that in the permutation language, this problem amounts to the calculation of the number of permutations with a *given* rise-and-descent sequence and has already been solved in terms of an elaborated combinatorial approach in [19–21]. Here we propose a novel solution of this problem.

Let  $X_l^{(n)}$  be a given trajectory generated by a random uniform permutation  $\pi$  of  $[n + 1]$ . This trajectory, according to the definition of the PGRW, can be uniquely defined in terms of the sequence of rises ( $\uparrow$ ) and descents ( $\downarrow$ ) characterizing the permutation  $\pi$ . Now, let  $\hat{I}_\uparrow$  and  $\hat{I}_\downarrow$  denote the integral operators of the form

$$\hat{I}_\uparrow = \int_x^1 dx \quad \text{and} \quad \hat{I}_\downarrow = \int_0^x dx. \quad (63)$$

Further on, to each  $l$ -step trajectory  $X_l^{(n)}$ , we associate a polynomial  $Q_{X_l^{(n)}}(x) \in \mathbb{R}[x]$  of degree  $l$ , defined as

$$Q_{X_l^{(n)}}(x) = \prod_{i=1}^l \hat{I}_i \cdot 1 = \hat{I}_1 \hat{I}_2 \hat{I}_3 \cdots \hat{I}_l \cdot 1 \quad (64)$$

where  $\hat{I}_i$  assumes either of two values,  $\hat{I}_\uparrow$  and  $\hat{I}_\downarrow$ , prescribed by the direction of the ‘arrow’ at the  $i$ th step in the corresponding sequence. In particular, the polynomial  $Q_{X_l^{(n)}}(x)$  corresponding to the  $l = 5$ -step trajectory  $X_l^{(n)} = \{\uparrow, \uparrow, \downarrow, \uparrow, \uparrow\}$  will be

$$\begin{aligned} Q_{X_l^{(n)}}(x) &= \hat{I}_\uparrow \hat{I}_\uparrow \hat{I}_\downarrow \hat{I}_\uparrow \hat{I}_\uparrow \cdot 1 \\ &= \int_x^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \int_{x_4}^1 dx_5 \cdot 1 \\ &= \frac{3}{40} - \frac{x}{8} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{120}. \end{aligned} \quad (65)$$

Then, the desired probability measure of a given trajectory  $X_l^{(n)}$  is given by

$$p(X_l^{(n)}) = \int_0^1 Q_{X_l^{(n)}}(x) dx = \int_0^1 dx \prod_{i=1}^l \hat{I}_i \cdot 1. \quad (66)$$

Note that this probability measure is not homogeneous, contrary to the measure of the standard Pólya walk.

The result in equation (66) may be compared with the analogous expression for the number of permutations with a given rise-and-descent sequence found by Niven [20]. Following Niven, consider a fixed up-and-down arrow sequence of length  $l$  and denote by  $k_1, k_2, \dots, k_r$  the positions of downarrows ( $r$  is the total number of downarrows) along the sequence. Suppose next that this up-and-down sequence corresponds to some random permutation  $\pi$  of  $[n + 1]$  such that, according to conventional notation, an *up*-arrow represents a rise, while a *down*-arrow is a descent. A question now is to calculate the number  $N(X_l^{(n)})$  of permutations generating a given up-and-down sequence (or, in our language, a given trajectory  $X_l^{(n)}$ ). Elaborated combinatorial arguments show that  $N(X_l^{(n)})$  equals the determinant of a matrix  $A$  of order  $r + 1$  whose elements  $\alpha_{i,j}$  (where  $i$  stands for the row, while  $j$  for the column) are binomial coefficients  $\binom{k_i}{k_{j-1}}$ , where  $k_0 = 0, k_{r+1} = l + 1$ , and  $\binom{m}{n} = 0$  if  $n > m$  [20]. Consequently, an alternative expression for the probability measure  $p(X_l^{(n)})$  may be written down as

$$p(X_l^{(n)}) = \frac{1}{(l + 1)!} \det \begin{pmatrix} 1 & 1 & \binom{k_1}{k_2} & \binom{k_1}{k_3} & \dots & \binom{k_1}{k_r} \\ 1 & \binom{k_2}{k_1} & 1 & \binom{k_2}{k_3} & \dots & \binom{k_2}{k_r} \\ 1 & \binom{k_3}{k_1} & \binom{k_3}{k_2} & 1 & \dots & \binom{k_3}{k_r} \\ & & & \dots & & \\ 1 & \binom{k_{r+1}}{k_1} & \binom{k_{r+1}}{k_2} & \binom{k_{r+1}}{k_3} & \dots & \binom{k_{r+1}}{k_r} \end{pmatrix}. \tag{67}$$

One may readily verify that both equations (66) and (67) reproduce our earlier results in equations (50) to (54) determining the probabilities of different four step trajectories.

Finally, similarly to our equations (20) and (42), we may define a general,  $k$ -point correlation function  $\mathcal{C}_{j_1, \dots, j_k}^{(k)}$  of the rise-and-descent sequence:

$$\mathcal{C}_{j_1, \dots, j_k}^{(k)} = \langle s_{j_1} s_{j_2} s_{j_3} \dots s_{j_k} \rangle \tag{68}$$

where the ‘spin’ variable  $s_k$  has been defined in equation (3). In terms of the auxiliary process  $Y_l$ ,  $\mathcal{C}_{j_1, \dots, j_k}^{(k)}$  can be rewritten formally as

$$\mathcal{C}_{j_1, \dots, j_k}^{(k)} = \left\langle \prod_{i=1}^k \theta(x_{j_{i+1}} - x_{j_i}) \right\rangle_{\{x_k\}} = \int_0^1 dx_{j_1} \dots \int_0^1 dx_{j_{k+1}} \prod_{i=1}^k \theta(x_{j_{i+1}} - x_{j_i}) \tag{69}$$

where the brackets with the subscript  $\{x_k\}$  denote averaging with respect to the distribution of the ensemble of variables  $x_k$ .

We note that the purely nearest-neighbours nature of the  $k$ -point correlations (which we have taken for granted in section 4) indeed becomes quite transparent for the process  $Y_l$ , such that  $\mathcal{C}_{j_1, \dots, j_k}^{(k)}$  factorizes automatically into a product of the corresponding correlation functions of the consecutive subsequences, in which all  $j_k$  differ by unity, as soon as any of the distances  $j_{k+1} - j_k$  exceeds unity. On the other hand, the correlation function  $\mathcal{C}^{(k)} = \mathcal{C}_{j, j+1, \dots, j+k}^{(k)}$  of a consecutive sequence of arbitrary order  $k$  can be obtained recursively using equation (69). To do this, we first note that  $\mathcal{C}^{(k)}$  can be represented as

$$\mathcal{C}^{(k)} = \int_0^1 dx f_k(x) \tag{70}$$

where  $f_k(x)$  are polynomials of order  $k$ , which obey the recursion

$$f_k(x) = - \int_0^x dx f_{k-1}(x) + \int_x^1 dx f_{k-1}(x) \quad f_0(x) = 1. \tag{71}$$

From this recursion, one finds immediately that  $\mathcal{C}^{(k)}$  are defined through

$$\mathcal{C}^{(k)} = \sum_{p=0}^{k-1} \frac{(-1)^k 2^k}{(k+1)!} \mathcal{C}^{(k-1-p)} \quad \mathcal{C}^{(0)} = 1 \quad (72)$$

such that the generating function of  $\mathcal{C}^{(k)}$  obeys

$$\mathcal{C}^{(k)}(z) \equiv \sum_{k=0}^{\infty} \mathcal{C}^{(k)} z^k = \frac{\tanh(z)}{z}. \quad (73)$$

Consequently,  $\mathcal{C}^{(k)}$  are related to the tangent numbers [22], and are given explicitly by

$$\mathcal{C}^{(k)} = \frac{(-1)^k 2^{k+2} (2^{k+2} - 1)}{(k+2)!} B_{k+2} \quad (74)$$

where  $B_k$  are the Bernoulli numbers. Note that since the Bernoulli numbers equal zero for  $k$  odd, we find that correlation functions of odd order vanish, i.e.  $\mathcal{C}^{(2k+1)} \equiv 0$ .

Finally, using the asymptotic expansions for the factorial and the Bernoulli numbers, we get that for  $k \gg 1$ , the  $k$ -point correlation functions  $\mathcal{C}^{(k)}$  ( $k$  is even) decay as

$$\mathcal{C}^{(k)} \sim 2(-1)^{k/2} \left(\frac{2}{\pi}\right)^{k+2} \quad (75)$$

i.e. the  $k$ -point correlations fall off exponentially with  $k$  with a characteristic length  $= \ln^{-1}(\pi/2)$ .

## 6. Distribution of the number of ‘turns’ of the PGRW trajectories

In this section we study an important measure of how scrambled the PGRW trajectories are. This measure is the number  $\mathcal{N}$  of ‘turns’ of an  $n$ -step PGRW trajectory, i.e. the number of times when the walker changes the direction of its motion. We focus here on the *asymptotic* form of the distribution  $\mathcal{P}(\mathcal{N}, n)$  of the number of ‘turns’ of an  $n$ -step PGRW trajectory.

In the permutation language, each turn to the left (right), when the walker making a jump to the right (left) at time moment  $j$  jumps to the left (right) at the next time moment  $j+1$  corresponds evidently to a peak  $\uparrow\downarrow$  (a through  $\downarrow\uparrow$ ) of a given permutation  $\pi$ , i.e. a sequence  $\pi_j < \pi_{j+1} > \pi_{j+2}$  ( $\pi_j > \pi_{j+1} < \pi_{j+2}$ ). Consequently, the distribution function  $\mathcal{P}(\mathcal{N}, n)$  of the number of ‘turns’ of the PGRW trajectory (i.e. the probability that an  $n$ -step PGRW trajectory has exactly  $\mathcal{N}$  turns) is just the distribution of the sum of peaks and throughs. The latter can be defined apparently using the so-called peak numbers  $P(n+1, m)$  of Stembridge (i.e., the number of permutations of  $[n+1]$  having  $m$  peaks), which obey the following three-term recurrence [23]:

$$P(n+1, m) = (2m+2)P(n, m) + (n+1-2m)P(n, m-1). \quad (76)$$

Below we will evaluate the asymptotic form of such a distribution function using a different type of arguments, other than the Stembridge formula in equation (76), expressing  $\mathcal{P}(\mathcal{N}, n)$  through the correlation functions of the rise-and-descent sequences.

Let  $\mathcal{N}_p$  and  $\mathcal{N}_t$  denote the number of peaks (i.e., the number of configurations  $\pi_j < \pi_{j+1} > \pi_{j+2}$ ) and throughs (i.e., the number of configurations  $\pi_j > \pi_{j+1} < \pi_{j+2}$ ) in a random permutation of  $[n+1]$ , respectively. These realization-dependent numbers can be written down explicitly as

$$\mathcal{N}_p = \frac{1}{4} \sum_{j=1}^{n-1} (1+s_j)(1-s_{j+1}) \quad (77)$$

and

$$\mathcal{N}_t = \frac{1}{4} \sum_{j=1}^{n-1} (1 - s_j)(1 + s_{j+1}) \tag{78}$$

where the ‘spin’ variable  $s_j$  has been defined in equation (3). The total number of turns  $\mathcal{N}$  of the RW trajectory, generated by a given permutation, is then given by

$$\mathcal{N} = \mathcal{N}_p + \mathcal{N}_t = \frac{1}{2} \sum_{j=1}^{n-1} (1 - s_j s_{j+1}). \tag{79}$$

We seek now the characteristic function  $\mathcal{Z}_n(k)$  of  $\mathcal{N}$ , defined as

$$\mathcal{Z}_n(k) = \langle \exp(ik\mathcal{N}) \rangle = \exp\left(\frac{ik(n-1)}{2}\right) \left\langle \exp\left(-\frac{ik}{2} \sum_{j=1}^{n-1} s_j s_{j+1}\right) \right\rangle \tag{80}$$

where the angle brackets denote averaging with respect to random permutations of  $[n + 1]$ . Note that  $\mathcal{Z}_n(k)$  in equation (80) can be thought of as a partition function of a somewhat exotic one-dimensional Ising-type model in which the ‘spin’ variables  $s_j$  of different sites are functionals of consecutive numbers in random permutations and thus possess rather specific correlations.

Now, since  $s_j s_{j+1} = \pm 1$ , the characteristic function  $\mathcal{Z}_n(k)$  in equation (80) can be written down as

$$\mathcal{Z}_n(k) = \left(\frac{1 + e^{ik}}{2}\right)^{n-1} \left\langle \prod_{j=1}^{n-1} (1 - t s_j s_{j+1}) \right\rangle \tag{81}$$

where  $t = \tanh(ik/2)$ . Averaging the product in equation (81) and taking into account that

- $s_j^2 \equiv 1$ ,
- $k$ -point correlations with  $k$  odd vanish,
- the  $k$ -point correlations factorize into a product of the correlation functions of corresponding subsequences, as soon as the distance between any two points exceeds unity,

we find that  $\mathcal{Z}_n(k)$  are polynomial functions of  $t$  of order  $[n/2]$ , where  $[x]$  is the floor function, of the following form

$$\mathcal{Z}_n(k) = \left(\frac{1 + e^{ik}}{2}\right)^{n-1} \sum_{l=0}^{[n/2]} (-1)^l W_{l,n} t^l \tag{82}$$

in which expression  $W_{l,n}$  denote the weights of corresponding configurations. These weights  $W_{l,n}$  can be obtained via straightforward combinatorial calculations and are given explicitly by

$$W_{l,n} = \sum \binom{n - 2l + 1}{\sum_{j=1}^l m_j} \frac{(\sum_{j=1}^l m_j)!}{m_1! m_2! m_3! \dots m_l!} \prod_{j=1}^l (C^{(2j)})^{m_j} \tag{83}$$

for  $l > 0$ , where the summation extends over all positive integers  $m_j$  obeying the equation:  $m_1 + 2m_2 + 3m_3 + \dots + lm_l = l$ , and

$$W_{l=0,n} = 1. \tag{84}$$

Note that in equation (83) the symbol  $\binom{a}{b}$ , in a standard fashion, denotes the binomial coefficient for  $a \geq b$  and equals zero otherwise.



Now, it is expedient to introduce the generating function of the form

$$\mathcal{Z}(k, z) = \sum_{n=2}^{\infty} \mathcal{Z}_n(k) z^n. \quad (85)$$

Note that summation over  $n$  is performed for  $n \geq 2$ , since  $\mathcal{N} \equiv 0$  for  $n = 1$  and  $n = 0$ . Multiplying both sides of equation (82) by  $z^n$  and performing summation, we get (relegating the intermediate steps to appendix B)

$$\begin{aligned} \mathcal{Z}(k, z) = & -\frac{4}{(1 + e^{ik})^2 z} - \frac{2}{1 + e^{ik}} - z \\ & + \frac{4}{(1 + e^{ik})^2 z} \left[ 1 - \frac{(1 + e^{ik})z}{2} \sum_{j=0}^{\infty} C^{(2j)} \left( \frac{(1 - e^{2ik})z^2}{4} \right)^j \right]^{-1}. \end{aligned} \quad (86)$$

Therefore, equation (86) relates the characteristic function of the sum of peaks and troughs in random permutations (or, equivalently, of the number of turns of the PGRW trajectories) to the  $k$ -point correlations in the rise-and-descent sequences, equation (74). Using next the expression in equation (73), we find, eventually, the following explicit result for the characteristic function  $\mathcal{Z}(k, z)$ :

$$\mathcal{Z}(k, z) = \frac{4}{(1 + e^{ik})^2 z} \left[ \left( \frac{1 - e^{ik}}{1 + e^{ik}} \right)^{1/2} \coth \left( (1 - e^{2ik})^{1/2} \frac{z}{2} \right) - 1 \right]^{-1} - \frac{2}{1 + e^{ik}} - z. \quad (87)$$

Expanding next  $\mathcal{Z}(k, z)$  in Taylor series in powers of variable  $k$ , we get

$$\mathcal{Z}(k, z) = \frac{z^2}{1 - z} + \frac{2z^2 ik}{3(1 - z)^2} - \frac{z^2(60 + 15z + 6z^2 - z^3)}{90(1 - z)^3} \frac{k^2}{2} + \mathcal{O}(k^3) \quad (88)$$

which implies, in particular, that the average and averaged squared number of turns of the PGRW trajectory are given by

$$\langle \mathcal{N} \rangle = \frac{2(n - 1)}{3} \quad (89)$$

and

$$\langle \mathcal{N}^2 \rangle = \frac{(5n^2 - 7n + 2)}{12} \theta(n - 1) + \frac{(n - 3)}{15} \theta(n - 3) + \frac{(n^2 - 7n + 12)}{36} \theta(n - 4). \quad (90)$$

Consider finally the form of  $\mathcal{Z}(k, z)$  in the limit of small  $k$  and  $z \rightarrow 1^-$ . In this limit, we find from equation (87) that in the leading order  $\mathcal{Z}(k, z)$  obeys

$$\mathcal{Z}(k, z) \sim \frac{1}{1 - z - \frac{2i}{3}k + \frac{4}{45}k^2}. \quad (91)$$

Inverting equation (91) with respect to  $k$  and  $z$ , we thus obtain that in the asymptotic limit  $n \rightarrow \infty$ , the distribution function  $\mathcal{P}(\mathcal{N}, n)$  of the number of ‘turns’ of the PGRW trajectory converges to a normal distribution of the form

$$\mathcal{P}(\mathcal{N}, n) \sim \frac{3}{4} \left( \frac{5}{\pi n} \right)^{1/2} \exp \left( -\frac{45(\mathcal{N} - \frac{2}{3}n)^2}{16n} \right). \quad (92)$$

This is not, of course, a counter-intuitive result in view of the polynomial three-term recursion in equation (76) obeyed by (closely related to  $\mathcal{P}(\mathcal{N}, n)$ ) peak numbers  $P(n + 1, m)$  [23].

### 7. Diffusion limit

Consider finally a continuous space and time version of the PGRW in the diffusion limit. To do this, it is expedient to define first some sort of an ‘evolution’ equation for  $\mathcal{P}_l(Y)$ .

We proceed as follows. Define first the polynomial

$$V^{(l)}(x, Y) = \sum_{Y_l=Y} Q_{Y_l}(x) \tag{93}$$

where the polynomial  $Q_{Y_l}(x)$  has been determined in equation (64) and the sum extends over all  $l$ -steps trajectories  $Y_l$  starting at zero and ending at the fixed point  $Y$ . Note next that one has

$$\mathcal{P}_l(Y) = \langle V^{(l)}(x, Y) \rangle_{\{x_l\}}. \tag{94}$$

Now, for the polynomials  $V^{(l)}(x, Y)$  one obtains, by counting all possible trajectories  $Y_l$  starting from zero and ending at the fixed point  $Y$ , the following ‘evolution’ equation:

$$V^{(l+1)}(x, Y) = \hat{I}_\uparrow \cdot V^{(l)}(x, Y - 1) + \hat{I}_\downarrow \cdot V^{(l)}(x, Y + 1). \tag{95}$$

Taking next advantage of the established equivalence between the processes  $Y_l$  and  $X_l^{(n)}$ , we can rewrite the last equation, upon averaging it over the distribution of variables  $\{x_l\}$ , as

$$\mathcal{P}_{l+1}(Y) = \frac{(l - Y + 4)}{2(l + 1)} \mathcal{P}_l(Y - 1) + \frac{(l + Y)}{2(l + 1)} \mathcal{P}_l(Y + 1) \tag{96}$$

which represents the desired evolution equation for  $\mathcal{P}_l(Y)$  in discrete space and time.

We hasten to remark that equation (96) can be thought of as a direct consequence of the celebrated relation between the Eulerian numbers:

$$\left\langle \begin{matrix} n + 1 \\ l \end{matrix} \right\rangle = l \left\langle \begin{matrix} n \\ l \end{matrix} \right\rangle + (n + 2 - l) \left\langle \begin{matrix} n \\ l - 1 \end{matrix} \right\rangle. \tag{97}$$

We emphasize, however, that despite the fact such a relation is evident for the distribution  $\mathcal{P}_l(X_l^{(l)} = X)$  of the end point of the PGRW trajectories, equation (11), its generalization to the distribution of the intermediate steps  $\mathcal{P}_l(X_l^{(n)} = X)$  as well as to the distribution of the process  $Y_l$ ,  $\mathcal{P}_l(Y = X)$ , is a completely uneventful *a priori* result, which enlightens the process under consideration.

Given the evolution equation (96), we turn next to the diffusion limit. Introducing space  $y = aY$  and time  $t = \tau l$  variables, where  $a$  and  $\tau$  define characteristic space and time scales, we turn to the limit  $a, \tau \rightarrow 0$ , supposing that the ratio  $a^2/\tau$  remains fixed and determines the diffusion coefficient  $D_0 = a^2/2\tau$ . In this limit, equation (96) becomes

$$\frac{\partial}{\partial t} \mathcal{P}(y, t) = \frac{\partial}{\partial y} \left( \frac{y}{t} \mathcal{P}(y, t) \right) + D_0 \frac{\partial^2}{\partial y^2} \mathcal{P}(y, t). \tag{98}$$

Note that the resulting continuous space and time equation is of the Fokker–Planck type; it has a constant diffusion coefficient and a negative drift term which, similarly to the Ornstein–Uhlenbeck process, grows linearly with  $y$ , but the amplitude of the drift decays in proportion to the first inverse power of time, which signifies that the process  $Y_l$  eventually delocalizes. Note also that the form of equation (98) ensures the conservation of the total probability.

The Green function solution of equation (98), remarkably, is a normal distribution

$$\mathcal{P}(y, t) = \sqrt{\frac{3}{4\pi D_0 t}} \exp\left(-\frac{3y^2}{4D_0 t}\right) \tag{99}$$

which is consistent with the large- $n$  limit of the discrete process derived in equation (18).

Now, the Langevin equation corresponding to equation (98) reads

$$\frac{dy}{dt} = -\frac{y}{t} + \zeta(t) \quad (100)$$

with  $\langle \zeta(t)\zeta(t') \rangle = 2D_0\delta(t-t')$ . Its solution can readily be obtained and has the following form

$$y(t) = \frac{1}{t} \int_0^t t' \zeta(t') dt'. \quad (101)$$

Note that equation (98) or (100) model dynamics of an overdamped particle in a one-dimensional continuum subject to a force  $-y/t$  and a white noise. Note also that equation (101) implies (see [24]) that  $y(t)$ , being a linear functional of a white noise, can be represented as a rescaled Brownian motion of the form

$$y(t) = \int_0^t \zeta'(t') dt' \quad (102)$$

where  $\langle \zeta'(t)\zeta'(t') \rangle = 2D_0\delta(t-t')/3$ . Consequently,  $y(t)$  can be also thought of as the solution of the customary Langevin equation with zero external force,

$$\frac{dy}{dt} = \zeta'(t). \quad (103)$$

We finally remark that equation (103) could have been derived directly from the definition of the discrete process  $Y_l$ , by taking the proper continuum limit of equation (55). The correlator  $\langle \zeta'(t)\zeta'(t') \rangle$  must be taken in this case as the limit of the continuation  $\tilde{C}^{(2)}(m)$  of  $C^{(2)}(m)$  to  $m < 0$ :

$$\langle \zeta'(t)\zeta'(t') \rangle = \lim_{a, \tau \rightarrow 0} \tilde{C}^{(2)}(m) = \lim_{a, \tau \rightarrow 0} (\delta_{m,0} - \frac{1}{3}(\delta_{m,-1} + \delta_{m,1})) = 2D_0\delta(t-t')/3. \quad (104)$$

The Gaussian nature of the process  $y(t)$  is transparent within this formalism.

## 8. Conclusions

In conclusion, we have studied here a simple model of a non-Markovian random walk, evolving in discrete time on a one-dimensional lattice of integers, in which the moves of the walker to the right or to the left are prescribed by the rise-and-descent sequence characterizing each random permutation  $\pi$  of  $[n+1] = \{1, 2, 3, \dots, n+1\}$ . We have determined exactly the probability  $\mathcal{P}_n(X)$  of finding the end-point  $X_n$  of the walker's trajectory at site  $X$ . Furthermore, we have shown that in the long-time limit  $\mathcal{P}_n(X)$  converges to a normal distribution in which an effective diffusion coefficient  $D = 1/6$  is three times smaller than the diffusion coefficient ( $D = 1/2$ ) of the conventional one-dimensional Pólya random walk. This implies that correlations in the generator of the walk, random permutations, which arise because only a finite amount of numbers is being shuffled and neither of these number may be equal to each other, are marginally important. Indeed, we have shown that two- and -four-point correlations in the sequence of rises depend only on the relative distance between them and extend effectively to nearest neighbours only. Next, at a closer look on the intrinsic features of the PGRW trajectories, we have formulated an auxiliary stochastic Markovian process. We have shown that despite the fact that this process is Markovian, while the random walk generated by permutations is not a Markovian process, the distribution of the auxiliary stochastic process appears to be identical to the distribution of the intermediate states of the walker's trajectories. This enabled us to obtain the probability measure of different excursions of the permutations-generated random walk, determine general  $k$ -point correlation functions and to evaluate the asymptotic form of the probability distribution of the number of turns in an  $n$ -step PGRW trajectory. Finally, we have discussed, in the diffusion limit, the continuous space and time version of such a walk.

**Acknowledgments**

The authors gratefully acknowledge helpful discussions with M N Popescu, S Dietrich and R Metzler. We also wish to thank S Nechaev for many valuable comments and for pointing us on the paper by Stembridge [23]. R V thanks Max-Planck Institute Stuttgart for warm hospitality and GO thanks the Alexander von Humboldt Foundation for the financial support via the Bessel Research Award.

**Appendix A**

Here we briefly outline the derivation of the integral representation of  $\mathcal{P}_n(X)$  in equation (15). Using the following expansion of the polylogarithm function [16]:

$$\text{Li}_{-n-1}(\exp(-2ik)) = \frac{(n + 1)!}{(2\pi i)^{n+2}} \sum_{q=-\infty}^{\infty} \frac{1}{\left(q + \frac{k}{\pi}\right)^{n+2}} \tag{105}$$

we have that  $\tilde{\mathcal{P}}_n(k)$  in equation (29) can be cast into the form

$$\tilde{\mathcal{P}}_n(k) = \frac{\sin^{n+2}(k)}{\pi^{n+2}} \sum_{q=-\infty}^{\infty} \frac{1}{\left(q + \frac{k}{\pi}\right)^{n+2}}. \tag{106}$$

Noting next that

$$\begin{aligned} \sum_{q=-\infty}^{\infty} \frac{1}{\left(q + \frac{k}{\pi}\right)^{n+2}} &= \frac{1}{(n + 1)!} \left[ \Psi_{n+1}\left(1 - \frac{k}{\pi}\right) + (-1)^n \Psi_{n+1}\left(\frac{k}{\pi}\right) \right] \\ &= \frac{(-1)^{n+1} \pi^{n+2}}{(n + 1)!} \frac{d^{n+1}}{dk^{n+1}} \cot(k) \end{aligned} \tag{107}$$

where  $\Psi_n(x)$  denotes the polygamma function [16], we arrive eventually at the representation in equation (15).

**Appendix B**

Here we present the details of the derivation of the generating function in equation (86). Multiplying both sides of equation (82) by  $z^n$  and performing summation, we get

$$\begin{aligned} \mathcal{Z}(k, z) &= \frac{2}{1 + e^{ik}} \sum_{l=0}^{\infty} \left(\frac{1 - e^{ik}}{1 + e^{ik}}\right)^l \sum_{n=2l}^{\infty} W_{l,n} \left(\frac{z(1 + e^{ik})}{2}\right)^n - \frac{2}{1 + e^{ik}} - z \\ &= \frac{2}{1 + e^{ik}} \sum_{l=0}^{\infty} \left(\frac{1 - e^{ik}}{1 + e^{ik}}\right)^l \left( \sum_{m_1!m_2!m_3! \dots m_l!} \frac{(\sum_{j=1}^l m_j)!}{m_1!m_2!m_3! \dots m_l!} \prod_{j=1}^l (C^{(2j)})^{m_j} \right) \\ &\quad \times \sum_{n=2l}^{\infty} \binom{n - 2l + 1}{\sum_{j=1}^l m_j} \left(\frac{z(1 + e^{ik})}{2}\right)^n - \frac{2}{1 + e^{ik}} - z \\ &= \frac{2}{1 + e^{ik}} \sum_{l=0}^{\infty} \left(\frac{1 - e^{2ik}}{4}\right)^l z^{2l} \left( \sum_{m_1!m_2!m_3! \dots m_l!} \frac{(\sum_{j=1}^l m_j)!}{m_1!m_2!m_3! \dots m_l!} \prod_{j=1}^l (C^{(2j)})^{m_j} \right) \\ &\quad \times \sum_{p=0}^{\infty} \binom{p + 1}{\sum_{j=1}^l m_j} \left(\frac{z(1 + e^{ik})}{2}\right)^p - \frac{2}{1 + e^{ik}} - z. \end{aligned} \tag{108}$$

Note now that  $\binom{p+1}{\sum_{j=1}^l m_j} \equiv 0$  for  $p < \sum_{j=1}^l m_j - 1$ . Hence, for  $l > 0$  (when, evidently,  $\sum_{j=1}^l m_j \geq 1$ ), we have

$$\begin{aligned} \sum_{p=0}^{\infty} \binom{p+1}{\sum_{j=1}^l m_j} \left( \frac{z(1+e^{ik})}{2} \right)^p &= \sum_{p=\sum_{j=1}^l m_j-1}^{\infty} \binom{p+1}{\sum_{j=1}^l m_j} \left( \frac{z(1+e^{ik})}{2} \right)^p \\ &= \left( \frac{z(1+e^{ik})}{2} \right)^{\sum_{j=1}^l m_j-1} \left( 1 - \frac{z(1+e^{ik})}{2} \right)^{-1-\sum_{j=1}^l m_j}. \end{aligned} \quad (109)$$

On the other hand, for  $l = 0$  (when  $\sum_j m_j \equiv 0$ ),

$$\sum_{p=0}^{\infty} \binom{p+1}{0} \left( \frac{z(1+e^{ik})}{2} \right)^p = \left( 1 - \frac{z(1+e^{ik})}{2} \right)^{-1}. \quad (110)$$

Making use next of equations (109) and (110), as well as of the identity

$$\sum_{l=1}^{\infty} \tau^l \sum \frac{(\sum_{j=1}^l m_j)!}{m_1! m_2! m_3! \cdots m_l!} \prod_{j=1}^l x_j^{m_j} = \frac{1}{1 - \sum_{j=1}^{\infty} x_j \tau^j} - 1 \quad (111)$$

we arrive eventually at the result in equation (86).

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